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Partner symmetries of the complex Monge–Ampère equation yield hyper-Kähler metrics without continuous symmetries

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Abstract

We extend the Mason–Newman Lax pair for the elliptic complex Monge–Ampère equation so that this equation itself emerges as an algebraic consequence. We regard the function in the extended Lax equations as a complex potential. Their differential compatibility condition coincides with the determining equation for the symmetries of the complex Monge–Ampère equation. We shall identify the real and imaginary parts of the potential, which we call partner symmetries, with the translational and dilatational symmetry characteristics, respectively. Then we choose the dilatational symmetry characteristic as the new unknown replacing the Kähler potential. This directly leads to a Legendre transformation. Studying the integrability conditions of the Legendre-transformed system we arrive at a set of linear equations satisfied by a single real potential. This enables us to construct non-invariant solutions of the Legendre transform of the complex Monge–Ampère equation. Using these solutions we obtained explicit Legendre-transformed hyper-Kähler metrics with a anti-self-dual Riemann curvature 2-form that admit no Killing vectors. They satisfy the Einstein field equations with Euclidean signature. We give the detailed derivation of the solution announced earlier and present a new solution with an added parameter. We compare our method of partner symmetries for finding non-invariant solutions to that of Dunajski and Mason who use 'hidden' symmetries for the same purpose.

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1. Introduction

In this paper, we shall present a method for finding non-invariant solutions of the elliptic complex Monge–Ampère equation, hereafter to be referred to as CMA_2,

$$u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{2}}u_{\bar{1}2} = 1 \tag{1.1}$$

using its symmetries in a non-standard way. This has application to important problems in physics and mathematics, in particular the instanton solutions of the Einstein field equations. They are described by four-dimensional Kähler metrics

$$\mathrm{d}s^2 = u_{i\bar{k}}\,\mathrm{d}z^i\,\mathrm{d}\bar{z}^k \tag{1.2}$$

where summation over the two values of both unbarred and barred indices is understood and subscripts denote partial derivatives. If the Kähler potential satisfies the elliptic complex Monge–Ampère equation, then the metric satisfies the vacuum Einstein field equations with Euclidean signature. We shall be interested in non-invariant solutions of CMA_2 which can be used to construct hyper-Kähler metrics without any Killing vectors. Among them is the $K3$ surface of Kummer which is the most important gravitational instanton [1].

Recently we suggested that group foliation can serve as a general method for finding non-invariant solutions of nonlinear partial differential equations [2, 3]. Historically this method goes back to the works of Lie [4] and Vessiot [5], see also Ovsiannikov [6] for a modern exposition. For CMA_2 group foliation was constructed in [2] but due to the complexity of the resolving equations non-invariant solutions have not yet been obtained in this way.

Therefore, in this paper we adopt a different approach which turned out to be fruitful specifically for CMA_2 which we shall call the method of *partner symmetries*, i.e. pairs of symmetries related by extended Lax equations. Using this method we obtain non-invariant solutions of the Legendre transform of (1.1). The earlier version of it was published in [7] where we derived linear partial differential equations for a class of non-invariant solutions of the hyperbolic CMA_2. The use of symmetries here is unlike their standard use in symmetry reduction [8] which leads to invariant solutions.

We start with the Lax equations [9, 10] appropriate to (1.1) for a nonlocal complex potential variable Φ. However, the commutator of the Lax operators does not reproduce CMA_2 itself, but only its differential consequences. In section 2 we supplement the Lax pair with another pair of linear equations so that in the extended linear system CMA_2 becomes an *algebraic* consequence. The crucial observation that follows is that this extended system has compatibility conditions which coincide with the determining equation for symmetries of CMA_2. The complex potential Φ will therefore be constructed from the symmetry characteristics [11] of CMA_2 which are inter-related by the extended Lax equations for real potentials given in section 3.

For the real part φ of the complex potential we shall use the translational symmetry of CMA_2 and the dilatational symmetry for its imaginary part ψ. In order to finally arrive at linear equations, having in mind that symmetry characteristics satisfy linear equations, we choose ψ as a new unknown instead of u which implies a Legendre transformation. This is given in section 4.

As a result we arrive at an over-determined set of second-order partial differential equations for the Legendre transform ψ of the unknown u. In section 5 we discuss its second- and third-order differential compatibility conditions.

In section 6 we integrate the third-order differential compatibility conditions of this system and arrive at six second-order equations satisfied by ψ with coefficients dependent on φ that has no compatibility conditions. In section 7 we choose the translational symmetry characteristic of φ. This leads to *linear* partial differential equations with constant coefficients for $v = \mathrm{e}^{-\psi}$. We obtain the general solution of these linear equations.

This solution is related to a particular solution set of the original equation (1.1) by a Legendre transformation and therefore determines a particular set of solutions of the Euclidean Einstein equations with anti-self-dual Riemann curvature 2-form. We shall not need to reconstruct the corresponding solution set of the original CMA_2 equation (1.1) by means of the inverse Legendre transformation, but instead make the Legendre transformation on the metric (1.2) itself. In section 8 we present the metric. Since the solution is obviously non-invariant, the corresponding metric has no Killing vectors. We discuss some properties of the metric. In particular, we show that it saturates Hitchin's bound [12] between the Euler number and Hirzebruch signature which means that if the manifold with our metric can be identified as a compact manifold it will coincide with the $K3$ surface, or a surface whose universal covering is $K3$. A preliminary version of this research with a less general set of solutions can be found in [13] and the announcement of those results is published in [14].

We discuss possible curvature singularities of our metric in section 9. We find that they coincide with those of the metric. We derive an additional first-order partial differential equation that the potential v must satisfy for the existence of singularities. We give the general solution for v that gives rise to singularities in the metric and curvature.

Finally, in section 10 we establish the relationship between our approach of partner symmetries to that developed recently by Dunajski and Mason [15, 16] who suggest invariance with respect to 'hidden' symmetries as a method for obtaining non-invariant solutions and apply it to Plebanski's second heavenly equation. For comparison, we construct nonlocal recursion operators for symmetries of CMA_2 and show that partnership between local symmetries of this equation is equivalent to the invariance of solutions of CMA_2 with respect to nonlocal symmetries of a very special form such that CMA_2 itself becomes a consequence of this invariance. The idea that invariance with respect to nonlocal 'potential' symmetries can give rise to non-invariant solutions of partial differential equations has appeared for the first time in the papers of Bluman and Kumei (see [17] and references therein).

2. Complex potential

In our approach we start with the Lax equations discovered by Mason and Newman [9, 10]

$$\Phi_1 = u_{1\bar{1}}\Phi_{\bar{2}} - u_{1\bar{2}}\Phi_{\bar{1}} \qquad \Phi_2 = u_{2\bar{1}}\Phi_{\bar{2}} - u_{2\bar{2}}\Phi_{\bar{1}} \tag{2.1}$$

where Φ is a complex-valued function of its arguments $\{z^i, \bar{z}^k\}$ and we skip the spectral parameter as unnecessary for our purposes. The commutator of the corresponding Lax pair

$$[\partial_1 + u_{1\bar{2}}\partial_{\bar{1}} - u_{1\bar{1}}\partial_{\bar{2}}, \partial_2 + u_{2\bar{2}}\partial_{\bar{1}} - u_{2\bar{1}}\partial_{\bar{2}}] = 0$$

does not reproduce CMA_2 but only its differential consequences resulting in the equation

$$u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{2}}u_{\bar{1}2} = k$$

where k is an arbitrary real constant with only three inequivalent choices $k = 1$, $k = -1$ and $k = 0$. Hence this Lax pair does not distinguish between elliptic, hyperbolic or homogeneous CMA_2 which is certainly its drawback.

Therefore, we supplement the Lax equations (2.1) with two more linear equations

$$\Phi_{\bar{1}} = u_{2\bar{1}}\Phi_1 - u_{1\bar{1}}\Phi_2 \qquad \Phi_{\bar{2}} = u_{2\bar{2}}\Phi_1 - u_{1\bar{2}}\Phi_2 \tag{2.2}$$

such that CMA_2 itself emerges as an algebraic compatibility condition of any three of these equations and also of the complex conjugate equations for $\bar{\Phi}$. Alternatively, if we impose

CMA_2 independently, then the additional pair of equations (2.2) follows from the original system (2.1) and CMA_2.

The differential compatibility condition of equations (2.1) taken in the form $(\Phi_1)_2 = (\Phi_2)_1$ and a similar condition for the complex conjugate system have the form of the determining equations for symmetry characteristics of CMA_2

$$\Box(\Phi) = 0 \qquad \Box(\bar{\Phi}) = 0 \tag{2.3}$$

where $\Box$ is the real linear differential operator

$$\Box = u_{2\bar{2}} D_1 D_{\bar{1}} + u_{1\bar{1}} D_2 D_{\bar{2}} - u_{2\bar{1}} D_1 D_{\bar{2}} - u_{1\bar{2}} D_2 D_{\bar{1}} \tag{2.4}$$

and similarly for the system (2.2) and its complex conjugate system. Here D_i are operators of total differentiation with respect to z^i.

The crucial consequence of (2.3) that we shall exploit in this paper is that the complex potential Φ will be constructed from the symmetry characteristics of CMA_2.

On the other hand, if we consider the second-order derivatives of the Kähler potential as unknowns in the system (2.1), (2.2), then its matrix has rank 3. Thus we may solve for the three derivatives

$$\begin{aligned} u_{1\bar{1}} &= \frac{\Phi_1 \Phi_{\bar{1}}}{\Phi_2 \Phi_{\bar{2}}} u_{2\bar{2}} + \frac{\Phi_1}{\Phi_{\bar{2}}} - \frac{\Phi_{\bar{1}}}{\Phi_2} \\ u_{1\bar{2}} &= \frac{\Phi_1}{\Phi_2} u_{2\bar{2}} - \frac{\Phi_{\bar{2}}}{\Phi_2} \qquad u_{2\bar{1}} = \frac{\Phi_{\bar{1}}}{\Phi_{\bar{2}}} u_{2\bar{2}} + \frac{\Phi_2}{\Phi_{\bar{2}}} \end{aligned} \tag{2.5}$$

and these expressions satisfy CMA_2 (1.1) identically. Substituting the expressions (2.5) into the equations complex conjugate to (2.1), (2.2) we obtain four equations with the only one unknown $u_{2\bar{2}}$ which give four different expressions for the same unknown. We need all these expressions to coincide which leads to a single algebraic compatibility condition

$$\Phi_1 \bar{\Phi}_2 - \Phi_2 \bar{\Phi}_1 = \bar{\Phi}_{\bar{1}} \Phi_{\bar{2}} - \bar{\Phi}_{\bar{2}} \Phi_{\bar{1}} \tag{2.6}$$

of the original linear system and its complex conjugate.

Finally, we find that the metric coefficients in (1.2) can be expressed through the complex potential

$$u_{i\bar{k}} = \frac{\Phi_i \bar{\Phi}_{\bar{k}} + \bar{\Phi}_i \Phi_{\bar{k}}}{\Phi_1 \bar{\Phi}_2 - \bar{\Phi}_1 \Phi_2} \tag{2.7}$$

which identically satisfy CMA_2 and reality conditions for u on account of (2.6). We can use (2.7) in the determining equations (2.3) for symmetries of CMA_2 in order to express them solely through the potentials Φ and $\bar{\Phi}$

$$\begin{aligned} &(\Phi_2 \bar{\Phi}_{\bar{2}} + \bar{\Phi}_2 \Phi_{\bar{2}}) \Phi_{1\bar{1}} + (\Phi_1 \bar{\Phi}_{\bar{1}} + \bar{\Phi}_1 \Phi_{\bar{1}}) \Phi_{2\bar{2}} - (\Phi_2 \bar{\Phi}_{\bar{1}} + \bar{\Phi}_2 \Phi_{\bar{1}}) \Phi_{1\bar{2}} \\ &\qquad - (\Phi_1 \bar{\Phi}_{\bar{2}} + \bar{\Phi}_1 \Phi_{\bar{2}}) \Phi_{2\bar{1}} = 0 \end{aligned} \tag{2.8}$$

and the complex conjugate equation. This is a system of coupled second-order quasi-linear equations. If we consider the equations (2.1), (2.2) and their complex conjugates as Bäcklund transformations, then the system of equations (2.6), (2.8) and the complex conjugate of the latter equation containing only potentials Φ, $\bar{\Phi}$ form the Bäcklund transform of the complex Monge–Ampère equation [7].

3. Partner symmetries

We shall work with real symmetry characteristics of CMA_2 which are the real and imaginary parts of the potential Φ. So we set

$$\Phi = \varphi + \mathrm{i}\psi$$

whereby (2.1), (2.2) and their complex conjugates become

$$\varphi_1 = \mathrm{i}(u_{1\bar{1}}\psi_{\bar{2}} - u_{1\bar{2}}\psi_{\bar{1}}) \qquad \varphi_2 = \mathrm{i}(u_{2\bar{1}}\psi_{\bar{2}} - u_{2\bar{2}}\psi_{\bar{1}}) \tag{3.1}$$

and

$$\psi_1 = -\mathrm{i}(u_{1\bar{1}}\varphi_{\bar{2}} - u_{1\bar{2}}\varphi_{\bar{1}}) \qquad \psi_2 = -\mathrm{i}(u_{2\bar{1}}\varphi_{\bar{2}} - u_{2\bar{2}}\varphi_{\bar{1}}) \tag{3.2}$$

together with their complex conjugate equations. CMA_2 is again an algebraic consequence of any three equations chosen from (3.1) and (3.2) while the system (3.2) follows from (3.1) plus CMA_2.

The differential compatibility conditions $(\varphi_1)_2 = (\varphi_2)_1$ and similarly for ψ are $\Box(\psi) = 0$ and $\Box(\varphi) = 0$ which again shows that φ and ψ are symmetry characteristics of CMA_2. As a consequence they will satisfy the nonlinear first-order compatibility condition

$$\psi_1\varphi_2 - \varphi_1\psi_2 = \varphi_{\bar{1}}\psi_{\bar{2}} - \psi_{\bar{1}}\varphi_{\bar{2}} \tag{3.3}$$

which is (2.6). We shall call any pair of symmetries φ and ψ related by equations (3.1) and (3.2) *partner symmetries* of CMA_2.

4. Legendre transformation and dilatational symmetry

We start with the general symmetry generator of CMA_2 [18]

$$X = \mathrm{i}(\Omega_1\partial_2 - \Omega_2\partial_1 - \Omega_{\bar{1}}\partial_{\bar{2}} + \Omega_{\bar{2}}\partial_{\bar{1}}) + C_1(z^1\partial_1 + \bar{z}^1\partial_{\bar{1}} + u\partial_u) + \mathrm{i}C_2(z^2\partial_2 - \bar{z}^2\partial_{\bar{2}}) + H\partial_u$$

where $\Omega(z^i, \bar{z}^k)$ and $H(z^i, \bar{z}^k)$ are arbitrary solutions of the linear system

$$\Omega_{1\bar{1}} = 0 \qquad \Omega_{2\bar{2}} = 0 \qquad \Omega_{1\bar{2}} = 0 \qquad \Omega_{2\bar{1}} = 0$$

and C_1 and C_2 are real constants. The corresponding symmetry characteristic [11] has the form

$$\widehat{\eta} = \mathrm{i}(u_1\Omega_2 - u_2\Omega_1 + u_{\bar{2}}\Omega_{\bar{1}} - u_{\bar{1}}\Omega_{\bar{2}}) + C_1(u - z^1u_1 - \bar{z}^1u_{\bar{1}}) - \mathrm{i}C_2(z^2u_2 - \bar{z}^2u_{\bar{2}}) + H \tag{4.1}$$

and φ, ψ can be chosen as special cases of expression (4.1). We note that the choice of φ and ψ as Ω and H respectively, leads to the metric for flat space and hence is trivial.

In the following we shall consider in detail the simplest non-trivial case when φ is identified with the symmetry characteristic independent of z^1, $\bar{z}^1$ and u_2, $u_{\bar{2}}$

$$\varphi = u_1\omega(z^2) + u_{\bar{1}}\bar{\omega}(\bar{z}^2) + h(z^2) + \bar{h}(\bar{z}^2) \tag{4.2}$$

which is a general form of such a symmetry following from (4.1). We shall find that in order to end up with linear equations with constant coefficients ω must be a constant and without loss of generality we may choose h as a linear function, so that (4.2) becomes

$$\varphi = u_1 + u_{\bar{1}} + \nu(z^2 + \bar{z}^2) \tag{4.3}$$

where ν is an arbitrary real constant.

We shall choose for ψ the characteristic of the dilatational symmetry

$$\psi = u - z^1u_1 - \bar{z}^1u_{\bar{1}} \tag{4.4}$$

and in order to arrive at linear equations we shall not substitute (4.3), (4.4) into the systems (3.1) and (3.2) directly. Instead, we regard ψ as a new unknown replacing the Kähler potential u. Then we recognize in formula (4.4) a part of the Legendre transformation

$$\begin{aligned} &\psi = u - z^1u_1 - \bar{z}^1u_{\bar{1}} \qquad u = \psi - p\psi_p - \bar{p}\psi_{\bar{p}} \\ &z^1 = -\psi_p \qquad \bar{z}^1 = -\psi_{\bar{p}} \qquad u_1 = p \qquad u_{\bar{1}} = \bar{p} \end{aligned} \tag{4.5}$$

where z^2 remains unchanged and ψ and φ are now regarded as functions of p, $\bar{p}$, z^2 and $\bar{z}^2$. This is therefore a partial Legendre transformation. We note that

$$\psi_{pp}\psi_{\bar{p}\bar{p}} - \psi_{p\bar{p}}^2 \neq 0 \tag{4.6}$$

is the existence condition for the Legendre transformation (4.5).

5. Legendre transform of the basic equations and their compatibility conditions

After the Legendre transformation (4.5) the system of eight basic equations (3.1), (3.2) and their complex conjugates is linear in the second derivatives of ψ and has rank five. Hence it can be solved with respect to five second derivatives ψ_{pp}, $\psi_{p\bar{2}}$, $\psi_{2\bar{2}}$, $\psi_{\bar{p}\bar{p}}$, $\psi_{\bar{p}2}$ in the form

$$\psi_{pp} = A\psi_{p\bar{p}} \qquad \psi_{p\bar{2}} = C\psi_{p\bar{p}} \qquad \psi_{2\bar{2}} = B\psi_{p\bar{p}} \tag{5.1}$$

together with their complex conjugates. The only remaining second derivative $\psi_{p\bar{p}}$ is regarded as parametric. In equations (5.1) the coefficients depend on the first derivatives of φ and ψ

$$\begin{aligned} A &= \frac{\varphi_p^2 + \psi_p^2 + \mathrm{i}(\varphi_p\psi_2 - \varphi_2\psi_p)}{\varphi_p\varphi_{\bar{p}} + \psi_p\psi_{\bar{p}}} \\ C &= \frac{\varphi_p\varphi_{\bar{2}} + \psi_p\psi_{\bar{2}} + \mathrm{i}(\varphi_{\bar{p}}\psi_p - \varphi_p\psi_{\bar{p}})}{\varphi_p\varphi_{\bar{p}} + \psi_p\psi_{\bar{p}}} \\ B &= \frac{\varphi_2\varphi_{\bar{2}} + \psi_2\psi_{\bar{2}} + \mathrm{i}(\psi_p\varphi_{\bar{2}} - \varphi_p\psi_{\bar{2}} + \psi_2\varphi_{\bar{p}} - \varphi_2\psi_{\bar{p}})}{\varphi_p\varphi_{\bar{p}} + \psi_p\psi_{\bar{p}}} \\ &= |A|^2 + |C|^2 - 1 \end{aligned} \tag{5.2}$$

where we have not yet made any choice of φ.

It is remarkable that two more second derivatives ψ_{p2} and $\psi_{\bar{p}\bar{2}}$ are cancelled in the equations (5.1) which allows us to end up with linear equations.

We note that after the Legendre transformation CMA_2 given by (1.1) takes the form

$$\psi_{p\bar{p}}\psi_{2\bar{2}} - \psi_{p\bar{2}}\psi_{\bar{p}2} = \psi_{pp}\psi_{\bar{p}\bar{p}} - \psi_{p\bar{p}}^2 \tag{5.3}$$

which is identically satisfied as a consequence of (5.1). Symmetry characteristic φ of CMA_2 (5.3) satisfies the determining equation which is a linearization of (5.3)

$$\psi_{\bar{p}\bar{p}}\varphi_{pp} + \psi_{pp}\varphi_{\bar{p}\bar{p}} - (2\psi_{p\bar{p}} + \psi_{2\bar{2}})\varphi_{p\bar{p}} + \psi_{\bar{p}2}\varphi_{p\bar{2}} + \psi_{p\bar{2}}\varphi_{\bar{p}2} - \psi_{p\bar{p}}\varphi_{2\bar{2}} = 0$$

or

$$\bar{A}\varphi_{pp} + A\varphi_{\bar{p}\bar{p}} - (B+2)\varphi_{p\bar{p}} + \bar{C}\varphi_{p\bar{2}} + C\varphi_{\bar{p}2} - \varphi_{2\bar{2}} = 0 \tag{5.4}$$

on account of equations (5.1).

The differential compatibility conditions for the system (5.1) have the form

$$(\psi_{pp})_{\bar{2}} = (\psi_{p\bar{2}})_p \qquad (\psi_{p\bar{2}})_2 = (\psi_{2\bar{2}})_p \tag{5.5}$$

and two complex conjugate conditions. They involve four third derivatives which are determined by differentiating (5.1)

$$\psi_{pp\bar{p}} = \frac{A\bar{A}_p + A_{\bar{p}}}{1-|A|^2}\psi_{p\bar{p}} \qquad \psi_{p\bar{p}2} = \left(\bar{C}\frac{A\bar{A}_p + A_{\bar{p}}}{1-|A|^2} + \bar{C}_p\right)\psi_{p\bar{p}} \tag{5.6}$$

and their complex conjugates. For example, we differentiate the first of the equations in (5.1) with respect to $\bar{p}$, use the complex conjugate equation and derive the equation for $\psi_{pp\bar{p}}$ as follows:

$$\begin{aligned} \psi_{pp\bar{p}} &= (\psi_{pp})_{\bar{p}} = (A\psi_{p\bar{p}})_{\bar{p}} = A(\psi_{\bar{p}\bar{p}})_p + A_{\bar{p}}\psi_{p\bar{p}} \\ &= A(\bar{A}\psi_{p\bar{p}})_p + A_{\bar{p}}\psi_{p\bar{p}} = |A|^2\psi_{pp\bar{p}} + (A\bar{A}_p + A_{\bar{p}})\psi_{p\bar{p}} \end{aligned}$$

and having determined the third derivatives in this way, we find that the compatibility conditions (5.5) result in

$$AC_{\bar{p}} - CA_{\bar{p}} - C_p + A_{\bar{2}} = 0 \qquad \bar{A}A_p + \bar{C}C_p - A_{\bar{p}} - C_2 = 0 \tag{5.7}$$

together with their complex conjugates.

6. Equations for potentials without compatibility conditions

In equation (4.2) we suggested the general form of the symmetry φ which has the Legendre transform

$$\varphi = p\omega(z^2) + \bar{p}\bar{\omega}(\bar{z}^2) + h(z^2) + \bar{h}(\bar{z}^2) \tag{6.1}$$

but so far we have not used it. Now we want to show that even keeping a more general form of φ, our goal of arriving at linear equations with constant coefficients fixes the final choice of φ as the Legendre transform of (4.3).

However, in order to make sure that φ will be a symmetry of the Legendre transform (5.3) of CMA_2 and to provide a symmetry between ψ and φ we shall impose on φ equations of the same form (5.1) as for ψ. We note that our particular choice (6.1) of φ will satisfy these equations identically and that this also agrees with the determining equation for symmetries of (5.3) since then equation (5.4) becomes the identity $|A|^2 + |C|^2 - B - 1 = 0$ in (5.2). As a consequence all our second-order compatibility conditions (5.7) are satisfied and the system (5.1) is compatible in the sense that the equations (5.5) are identically satisfied and the same equations for φ are satisfied as well.

Thus, for ψ we have the system of the second-order equations (5.1) and the third-order equations (5.6). The coefficients of the latter equations are logarithmic derivatives of $\varphi_p\varphi_{\bar{p}} + \psi_p\psi_{\bar{p}}$, so that these equations take the form

$$\frac{(\psi_{p\bar{p}})_p}{\psi_{p\bar{p}}} = \frac{A\bar{A}_p + A_{\bar{p}}}{1-|A|^2} = \frac{(\varphi_p\varphi_{\bar{p}} + \psi_p\psi_{\bar{p}})_p}{\varphi_p\varphi_{\bar{p}} + \psi_p\psi_{\bar{p}}}$$

$$\frac{(\psi_{p\bar{p}})_2}{\psi_{p\bar{p}}} = \bar{C}\frac{A\bar{A}_p + A_{\bar{p}}}{1-|A|^2} + \bar{C}_p = \frac{(\varphi_p\varphi_{\bar{p}} + \psi_p\psi_{\bar{p}})_2}{\varphi_p\varphi_{\bar{p}} + \psi_p\psi_{\bar{p}}}$$

and the integrated equations become an additional second-order equation

$$\psi_{p\bar{p}} = C_1(\varphi_p\varphi_{\bar{p}} + \psi_p\psi_{\bar{p}}) \tag{6.2}$$

equivalent to the third-order equations (5.6). Here C_1 is a real integration constant. In the case φ is not fixed, these equations hold for φ as well where we exchange φ to ψ and C_1 to C_2.

Hence ψ satisfies the system of six equations (5.1), (6.2)

$$\begin{aligned}
\psi_{p\bar{p}} &= \varphi_p\varphi_{\bar{p}} + \psi_p\psi_{\bar{p}} \\
\psi_{pp} &= \varphi_p^2 + \psi_p^2 + \mathrm{i}(\varphi_p\psi_2 - \varphi_2\psi_p) \\
\psi_{p\bar{2}} &= \varphi_p\varphi_{\bar{2}} + \psi_p\psi_{\bar{2}} + \mathrm{i}(\varphi_{\bar{p}}\psi_p - \varphi_p\psi_{\bar{p}}) \\
\psi_{2\bar{2}} &= \varphi_2\varphi_{\bar{2}} + \psi_2\psi_{\bar{2}} + \mathrm{i}(\psi_p\varphi_{\bar{2}} - \varphi_p\psi_{\bar{2}} + \psi_2\varphi_{\bar{p}} - \varphi_2\psi_{\bar{p}})
\end{aligned} \tag{6.3}$$

together with two complex conjugate equations and similarly for φ. Here we have used (6.2) in the system of equations (5.1) and the expressions (5.2) for the coefficients A, B and C and made the change of notation $C_1\psi \mapsto \psi$, $C_1\varphi \mapsto \varphi$. The differential compatibility conditions of these systems are identically satisfied without generating any second-, or third-order conditions. Thus the Legendre transform of the determining equation (2.3) for symmetries of CMA_2 is also identically satisfied for ψ and φ together with the first-order compatibility condition (3.3). Therefore, any φ and ψ which satisfy the systems (6.3) and the one obtained from (6.3) by the exchange of ψ and φ form a pair of partner symmetry characteristics related to each other by the Legendre transform of (3.1) and (3.2).

7. Linear equations and their general solution

The linearization is achieved by the logarithmic substitution

$$\psi = -\log v \tag{7.1}$$

so that the first equation in (6.3)

$$-\frac{v_{p\bar{p}}}{v} + \frac{v_p v_{\bar{p}}}{v^2} = \varphi_p \varphi_{\bar{p}} + \frac{v_p v_{\bar{p}}}{v^2}$$

becomes the linear equation

$$v_{p\bar{p}} = -\varphi_p \varphi_{\bar{p}} v \tag{7.2}$$

because according to our choice (4.2) φ is independent of $z^1, \bar{z}^1$ and $u_2, u_{\bar{2}}$. Indeed if we had admitted such dependence, then after the Legendre transformation we would have $z^1 = -\psi_p, u_2 = \psi_2$ so that the derivatives of φ in (7.2) and (6.3) would result in nonlinear dependence on v. Furthermore, the second derivatives $v_{p2}, v_{\bar{p}\bar{2}}$ would enter destroying the whole structure.

The linear equation (7.2) will have a constant coefficient if we further impose the condition $\varphi_p = \text{const}$. That is, we choose the symmetry φ to be linear in $p, \bar{p}$ which results in the Legendre transform of formula (4.3)

$$\varphi = p + \bar{p} + \nu(z^2 + \bar{z}^2) \tag{7.3}$$

where ν is an arbitrary real constant and our choice of $h(z^2) = \nu z^2$ in (4.2) has been made without loss of generality. Thus, from now on we fix the choice of the potential φ as the characteristic of translational symmetry (4.3) with its Legendre transform (7.3).

Then the equations (6.3) for $v = \mathrm{e}^{-\psi}$ become linear with constant coefficients

$$\begin{aligned}
&v_{p\bar{p}} + v = 0\\
&v_{pp} + v - \mathrm{i}(v_2 - \nu v_p) = 0\\
&v_{p\bar{2}} + \nu v - \mathrm{i}(v_p - v_{\bar{p}}) = 0\\
&v_{2\bar{2}} + \nu^2 v - \mathrm{i}[v_2 - v_{\bar{2}} + \nu(v_p - v_{\bar{p}})] = 0
\end{aligned} \tag{7.4}$$

plus two complex conjugate equations.

The general solution of the linear system (7.4) in the case of the discrete spectrum has the form

$$\begin{aligned}
v = \sum_{j=-\infty}^{\infty} &\exp\left\{2\,\mathrm{Im}\left(\left[\alpha_j^2\left(s_j^2+1\right)+1\right]z^2\right)\right\}\left\{\exp[2s_j\,\mathrm{Re}[\alpha_j(p+\nu z^2)]]\right.\\
&\times \mathrm{Re}\left\{F_j \exp\left[2\mathrm{i}\left[\mathrm{Im}(\alpha_j(p+\nu z^2)) - 2s_j\,\mathrm{Re}\left(\alpha_j^2 z^2\right)\right]\right]\right\}\\
&+ \exp[-2s_j\,\mathrm{Re}[\alpha_j(p+\nu z^2)]]\,\mathrm{Re}\left\{G_j \exp\left[2\mathrm{i}\left[\mathrm{Im}(\alpha_j(p+\nu z^2))\right.\right.\right.\\
&\left.\left.\left.\left.+2s_j\,\mathrm{Re}\left(\alpha_j^2 z^2\right)\right]\right]\right\}\right\}
\end{aligned} \tag{7.5}$$

where α_j, F_j, G_j are arbitrary complex constants and $s_j = \sqrt{1 - 1/|\alpha_j|^2}$. For a particular example, one can take a finite sum instead of infinite series in this formula. On the other hand we can also consider the continuous spectrum where α_j should be changed to α, s_j to $s = \sqrt{1 - 1/|\alpha|^2}$, F_j, G_j to $F(\alpha, \bar{\alpha}), G(\alpha, \bar{\alpha})$, respectively, and the sum in formula (7.5) should be changed to a double integral with respect to $\alpha, \bar{\alpha}$. In this way we end up with an integral representation of the solution for v. Since the solution (7.5) explicitly depends on four real independent variables, it is a non-invariant solution of the Legendre transform of CMA_2 given by (5.3).

For our particular purpose of constructing the metric without any Killing vectors we have no need to reconstruct the corresponding non-invariant solution of the original field equation (1.1), which is CMA_2 itself, by the inverse Legendre transformation. Instead we make the Legendre transformation (4.5) in the metric (1.2) itself.

8. The metric

The solution (7.5) of the linear equations (7.4) will be used in the construction of the metric which is a Legendre transform of the metric (1.2) . For this purpose it will be convenient to introduce a new notation for the numerators and the denominator of the coefficients A and C in (5.2) putting $A = a/c$, $C = \bar{b}/c$

$$\begin{aligned} a &= v^2 + v_p^2 - \mathrm{i}v(v_2 - \nu v_p) \\ b &= v_{\bar{p}}v_2 + \nu v^2 - \mathrm{i}v(v_p - v_{\bar{p}}) \\ c &= v^2 + |v_p|^2 \end{aligned} \tag{8.1}$$

so that the metric is given by

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{v^2(c^2 - |a|^2)} \bigg[& a(c\,\mathrm{d}p + b\,\mathrm{d}z^2)^2 + \bar{a}(c\,\mathrm{d}\bar{p} + \bar{b}\,\mathrm{d}\bar{z}^2)^2 \\ & + \frac{1}{c}(c^2 + |a|^2)|c\,\mathrm{d}p + b\,\mathrm{d}z^2|^2 \bigg] + \frac{1}{v^2 c}(c^2 - |a|^2)|\mathrm{d}z^2|^2 \end{aligned} \tag{8.2}$$

with the real potential v determined by (7.5). The solution (7.5) depends on four independent variables, so that it is a non-invariant solution. The Legendre-transformed hyper-Kähler metric (8.2) therefore has no Killing vectors. This general result is violated in only one case, if the sum in (7.5) is restricted to only one term. We note that the metric coefficients depend only on logarithmic derivatives of v and therefore in this special case the dependence on the argument of the first exponential factor in (7.5) vanishes and the metric depends on only three coordinates. This is a symmetry reduction.

Any solution for v of the form (7.5) with a minimum of two terms in the sum when substituted into (8.1) gives us an explicit form of the metric (8.2) without any Killing vectors.

It will be useful to express the metric (8.2) in the Euclidean Newman–Penrose formalism [19, 20]. The metric is given by

$$\mathrm{d}s^2 = l \otimes \bar{l} + \bar{l} \otimes l + m \otimes \bar{m} + \bar{m} \otimes m \tag{8.3}$$

where

$$\begin{aligned} l &= \frac{1}{v[c(c^2 - |a|^2)]^{1/2}}[c(c\,\mathrm{d}p + b\,\mathrm{d}z^2) + \bar{a}(c\,\mathrm{d}\bar{p} + \bar{b}\,\mathrm{d}\bar{z}^2)], \\ m &= \frac{(c^2 - |a|^2)^{1/2}}{vc^{1/2}}\,\mathrm{d}z^2 \end{aligned} \tag{8.4}$$

and the co-frame will be labelled as $\omega^a = \{l, \bar{l}, m, \bar{m}\}$.

It can be verified directly that in the Newman–Penrose frame with the metric coefficients given by (8.1) and the potential v satisfying the linear system (7.4), the Riemann curvature 2-form is anti-self-dual

$$\Omega^a{}_b = -{}^*\Omega^a{}_b \qquad \Omega^a{}_b = \tfrac{1}{2}R^a{}_{bcd}\,\omega^c \wedge \omega^d \tag{8.5}$$

where * is the Hodge star operator. Ricci-flatness follows by virtue of the first Bianchi identity. The metric (8.2) has no Killing vectors since the potential v in the solution (7.5) depends on all four coordinates. Its first Chern class vanishes since $R_{ik}\omega^i \wedge \omega^k = 0$. There are three real closed 2-forms that follow from the metric (8.2)

$$\theta_0 = l \wedge \bar{l} - m \wedge \bar{m} \qquad \theta_+ = \frac{1}{2}\left(l \wedge \bar{m} + \bar{l} \wedge m\right) \qquad \theta_- = \frac{1}{2\mathrm{i}}(l \wedge \bar{m} - \bar{l} \wedge m) \tag{8.6}$$

which shows that it is hyper-Kähler because the (1, 1) tensors that define the structure functions of three almost complex structures are obtained by raising an index of the 2-forms (8.6) with the metric [20].

We note that due to the fact that v is given by exponentials and the metric coefficients are homogeneous of degree zero in v and its derivatives, the metric coefficients tend to constant values asymptotically. The same is true for the Newman–Penrose tetrad scalars of the Riemann tensor.

We shall not discuss whether or not our solution describes a compact 4-manifold. All our analysis has been local and given a metric in a local coordinate chart as in (8.2), compactness is always an open question. The property of compactness depends on the range of coordinates that we may assign to the local coordinates $\{p, \bar{p}, z^2, \bar{z}^2\}$. We have not done that, but the presence of exponentials in the metric coefficients suggests that the metric could well be made compact by choosing a suitable domain of coordinates. Assuming compactness, by virtue of the anti-self-dual curvature property, our solution saturates Hitchin's bound $|\tau| \leqslant (2/3)\chi$ [12] between the Euler characteristic χ and the Hirzebruch signature τ. They are defined as integrals of the following 4-forms over a compact manifold

$$\chi = \frac{1}{2^4\pi^2}\int \Omega^a{}_b \wedge {}^*\Omega^b{}_a \qquad \tau = \frac{1}{24\pi^2}\int \Omega^a{}_b \wedge \Omega^b{}_a$$

and hence the quantity

$$\chi + \frac{3}{2}\tau = \frac{1}{2^4\pi^2}\int \Omega^a{}_b \wedge \left({}^*\Omega^b{}_a + \Omega^b{}_a\right) = 0$$

vanishes due to the anti-self-duality (8.5) of the Riemann curvature 2-form. Thus we have the saturation of the Hitchin bound. Only $K3$ and surfaces whose universal covering is $K3$ have this property according to Hitchin's theorem [12].

The metric (8.2) contains an infinite number of arbitrary parameters, even arbitrary functions of two variables in the double integral. Furthermore, it is an open question as to whether or not for suitably chosen values of the arbitrary constants and domain of coordinates this metric will be that of a compact manifold. This is an important issue, because of its relevance to the metric on $K3$. We know from index theorems [21] that the number of parameters in the metric for $K3$ is finite. Further work will possibly enable us to identify the essential parameters in this solution that actually characterize $K3$.

9. Analysis of possible curvature singularities

In order to discuss singularities in the curvature scalars we will need the tetrad scalars of the Riemann tensor in the Newman–Penrose frame (8.4). The Newman–Penrose tetrad scalars of the Riemann tensor are too lengthy to be presented here. However, their denominators are quite simple and the only places where curvature scalars blow up are given by

$$|a|^2 - c^2 = 0 \tag{9.1}$$

which are also singularities of the metric. Using (8.1) we arrive at the following first-order partial differential equation:

$$v[(v_p - v_{\bar{p}})^2 + |v_2 - \nu v_p|^2] + 2\,\mathrm{Im}\big[(v_2 - \nu v_p)\big(v^2 + v_{\bar{p}}^2\big)\big] = 0 \tag{9.2}$$

for possible curvature singularities. If for some v equation (9.2) is satisfied together with the system (7.4), then the denominator of the metric is identically zero and hence such a solution for v cannot be used for constructing the metric. From the definitions (8.1) and the singularity condition (9.1) we note the following relations:

$$c_p = \mathrm{i}(b - \nu c) \qquad a_{\bar{p}} = -\mathrm{i}(b - \nu c) \qquad a_p = \mathrm{i}\frac{a}{c}(b - \nu c) \tag{9.3}$$

$$\frac{b}{c} - \mathrm{i}\left(\frac{v_{\bar{p}}}{v}\frac{a}{c} - \frac{v_p}{v}\right) - \nu = 0 \qquad \frac{\bar{b}}{c} - \nu = \frac{c}{a}\left(\frac{b}{c} - \nu\right) \tag{9.4}$$

which are used to check that $A_p = A_{\bar{p}} = 0$, $C_p = C_{\bar{p}} = 0$.

Next we express v_{p2} from the last equation (9.3) and check the relation

$$\mathrm{i}\left(v_{p2} - v_{\bar{p}2}\frac{a}{c}\right) + v_2\left(\frac{b}{c} - \nu\right) = 0 \tag{9.5}$$

by differentiating the first of equations (9.4) with respect to z^2 and use the relation (9.5). We find

$$\left(\frac{b}{c}\right)_2 - \mathrm{i}\frac{v_{\bar{p}}}{v}\left(\frac{a}{c}\right)_2 = 0 \tag{9.6}$$

and differentiating this equation with respect to p, using the independence of $A = a/c$ and $\bar{C} = b/c$ from p we prove their independence of $z^2, \bar{z}^2$ as well by checking that $A_2 = A_{\bar{2}} = 0$, $C_2 = C_{\bar{2}} = 0$ and hence A and C are constants

$$\begin{aligned} &A = \frac{a}{c} = \lambda = \mathrm{const} \qquad \bar{C} = \frac{b}{c} = \mu = \mathrm{const} \\ &\bar{\lambda} = \frac{1}{\lambda} \qquad \bar{\mu} = \frac{\mu + (\lambda - 1)\nu}{\lambda} \end{aligned} \tag{9.7}$$

which serve as the definition of λ and μ. From the definitions of a and c it follows that

$$v(c - a) + \mathrm{i}v_p(b - \nu c) - \mathrm{i}(v_2 - \nu v_p)c = 0$$

which due to (9.7) becomes

$$v_2 = \mu v_p + \mathrm{i}(\lambda - 1)v \tag{9.8}$$

and relation (9.5) takes the form

$$v_p = \lambda v_{\bar{p}} + \mathrm{i}(\mu - \nu)v. \tag{9.9}$$

Compatibility conditions $(v_p)_2 = (v_2)_p$ and $(v_p)_{\bar{2}} = (v_{\bar{2}})_p$ of (9.9), (9.8) and the complex conjugate to (9.8) are identically satisfied.

Consecutive integration of these three first-order equations gives the solution

$$\begin{aligned} &v = \exp\left\{\mathrm{i}(\lambda - 1)\left(z^2 + \frac{1}{\lambda}\bar{z}^2\right) + \mathrm{i}(\mu - \nu)\xi\right\}H(\eta) \\ &\xi = p + \mu z^2 \qquad \eta = \lambda\xi + \bar{\xi} \end{aligned} \tag{9.10}$$

Due to the first of the second-order equations (7.4) the function $H(\eta)$ satisfies the following ordinary differential equation:

$$\lambda H'' + \mathrm{i}(\mu - \nu)H' + H = 0 \tag{9.11}$$

with the general solution

$$\begin{aligned} H = {} & C_1 \exp\left\{\frac{\mathrm{i}}{2\lambda}\left[-(\mu - \nu) + \sqrt{(\mu - \nu)^2 + 4\lambda}\right]\eta\right\} \\ & + \bar{C}_1 \exp\left\{-\frac{\mathrm{i}}{2\lambda}\left[\mu - \nu + \sqrt{(\mu - \nu)^2 + 4\lambda}\right]\eta\right\} \end{aligned} \tag{9.12}$$

where C_1, $\bar{C}_1$ are arbitrary constants and the second constant was chosen as $\bar{C}_1$ to satisfy the reality condition

$$e^{i(\mu-\nu)\xi} H = e^{-i\frac{(\mu-\nu)}{\lambda}\bar{\xi}} \bar{H}$$

for the solution v defined by (9.10). In the notation

$$\lambda = -\frac{\alpha}{\bar{\alpha}} \qquad \mu = \nu - 2\,\mathrm{i}\alpha s \qquad s = \sqrt{1 - \frac{1}{|\alpha|^2}} \qquad C_1 = \bar{F}$$

the singular solution (9.10) with H defined by (9.12) finally becomes

$$\begin{aligned} v = \mathrm{Re}\{F \exp\{&\alpha(s+1)p + \bar{\alpha}(s-1)\bar{p} - \mathrm{i}[\alpha^2(s+1)^2 + 1 \\ &+ \mathrm{i}\nu\alpha(s+1)]z^2 + \mathrm{i}[\bar{\alpha}^2(s-1)^2 + 1 - \mathrm{i}\nu\bar{\alpha}(s-1)]\bar{z}^2\}\} \end{aligned} \tag{9.13}$$

where F is an arbitrary complex constant. The expression (9.13) for v is the general solution of the system of linear second-order equations (7.4) supplemented by the first-order singularity condition (9.2). It is obvious that the singular solution (9.13) is a very special case of the general solution (7.5) which is obtained if we restrict ourselves to only one term in the sum (7.5) and keep only one complex constant F putting the other G equal to zero. This solution is automatically avoided since the minimum number of terms in the sum (7.5) is two in order not to have symmetry reduction.

Thus, we have proved that any solution (7.5) of the linear system (7.4) which does not coincide with (9.13) can be used to construct the explicit expression for metric (8.2).

10. Recursion operators and invariance with respect to nonlocal symmetries

In this section, we establish the relationship between our use of partner symmetries for obtaining non-invariant solutions of CMA_2 and the method of Dunajski and Mason [15, 16]. Dunajski and Mason have suggested a general approach for finding non-invariant solutions as invariant solutions with respect to 'hidden' symmetries and applied it to the second heavenly equation of Plebanski.

We introduce a pair of linear differential operators

$$L_1 = \mathrm{i}(u_{1\bar{2}}D_{\bar{1}} - u_{1\bar{1}}D_{\bar{2}}) \qquad L_2 = \mathrm{i}(u_{2\bar{2}}D_{\bar{1}} - u_{2\bar{1}}D_{\bar{2}}) \tag{10.1}$$

plus their complex conjugates. These operators commute as a consequence of CMA_2, alternatively $[L_1, L_2] = 0$ implies differential consequences of CMA_2. In terms of these operators the relations (3.1) and (3.2) between potentials take the form

$$\begin{aligned} \varphi_1 &= -L_1\psi \qquad & \varphi_2 &= -L_2\psi \\ \psi_1 &= L_1\varphi \qquad & \psi_2 &= L_2\varphi \end{aligned} \tag{10.2}$$

The determining equation $\Box(\varphi) = 0$ for symmetries of CMA_2 with the operator $\Box$ given by (2.4) can be expressed in terms of the operators (10.1)

$$L_2D_1\varphi = L_1D_2\varphi \quad \Longleftrightarrow \quad D_1L_2\varphi = D_2L_1\varphi \tag{10.3}$$

Equation (10.3) is a conservation law which implies the existence of a potential ψ for the symmetry φ such that

$$D_1\psi = L_1\varphi \qquad D_2\psi = L_2\varphi \tag{10.4}$$

and we note that ψ satisfies the same equation $\Box(\psi) = 0$ taken in the form (10.3)

$$D_1L_2\psi = D_2L_1\psi \quad \Longrightarrow \quad L_2D_1\psi = L_1D_2\psi \quad \Longrightarrow \quad L_2L_1\varphi = L_1L_2\varphi$$

where we have used equations (10.4). The last equation is satisfied identically since the operators L_1 and L_2 commute on the solution manifold of CMA_2.

Hence we find that if φ is a symmetry of CMA_2 then ψ defined by

$$\psi = D_1^{-1} L_1 \varphi = D_2^{-1} L_2 \varphi \tag{10.5}$$

is also a symmetry. Therefore, the integro-differential operators

$$R_1 = D_1^{-1} L_1 \qquad R_2 = D_2^{-1} L_2 \tag{10.6}$$

are recursion operators for symmetries of CMA_2 defined on the subspace of symmetries φ satisfying the relation $R_1\varphi = R_2\varphi$.

The relations (10.2) between the potentials become

$$\varphi = -R_1\psi \qquad \varphi = -R_2\psi \qquad \psi = R_1\varphi \qquad \psi = R_2\varphi \tag{10.7}$$

so that comparison of (10.5) with the two latter formulae in (10.7) shows that our second symmetry ψ which is a partner for φ coincides with the potential for φ. Hence the recursion operators R_i are defined on the subspace of partner symmetries of CMA_2.

The operators R_i are recursion operators for the whole space of symmetries of CMA_2 if we define the inverse integral operators D_i^{-1} by specifying the limits of integration. The relations $D_i^{-1} D_i = 1$ for $i = 1, 2$ will be satisfied if we define $D_i^{-1} = \int_a^{z^i} \mathrm{d}z^i$ and impose the boundary conditions $f(a) = 0$ for all f in the domain of definition of D_i^{-1}. In particular, if $a = \infty$ we need to restrict ourselves to the space of functions decreasing to zero when z^i tend to infinity. Then we have

$$D_i^{-1} D_i f = \int_a^{z^i} D_i(f(z^i))\, \mathrm{d}z^i = f(z^i)$$

for all such f and hence $D_i^{-1} D_i = 1$.

Recursion operators R_i should commute with the operator (2.4) of the determining equation for symmetries of CMA_2 on the space of its solutions on account of CMA_2. Calculating these commutators we obtain

$$[R_i, \Box] = \left[L_i, D_i^{-1}\right] \Box$$

where

$$\Box = \mathrm{i}(D_2 L_1 - D_1 L_2)$$

coincides with the operator (2.4). Hence on the space of solutions of the equation $\Box(\varphi) = 0$ we have the required property $[R_i, \Box]\varphi = 0$.

If we choose for φ and ψ *any* local symmetries of CMA_2, then the following expressions are characteristics of nonlocal symmetries

$$\widehat{\eta}_1 = R_1\psi + \varphi \qquad \widehat{\eta}_2 = R_2\psi + \varphi \qquad \widehat{\eta}_3 = R_1\varphi - \psi \qquad \widehat{\eta}_4 = R_2\varphi - \psi \tag{10.8}$$

so that the relations (10.7) of partnership between local symmetries take the form of equations determining invariant solutions of CMA_2 with respect to the nonlocal symmetries (10.8)

$$\widehat{\eta}_i = 0 \qquad i = 1, 2, 3, 4 \tag{10.9}$$

of a very particular type (10.8). Taking into account CMA_2, only two relations out of the system (3.1), (3.2) are independent and therefore the same is true for the invariance conditions (10.9). Alternatively, CMA_2 itself is an algebraic consequence of the invariance conditions (10.9). We note that the definition of D_i^{-1} so that R_i become recursion operators on the space of all symmetries of CMA_2 is not needed after we impose the condition (10.9) because it ensures that R_i now act only on the subspace of partner symmetries.

Thus, the starting point of our use of local partner symmetries of CMA_2 is equivalent to searching for its solutions invariant with respect to two nonlocal symmetries of a special form (10.8) which is in the spirit of Dunajski and Mason's treatment of Plebanski's second heavenly equation. Both of these approaches exploit the idea that an appropriate non-standard use of symmetries of a nonlinear partial differential equation may result in a construction of its non-invariant solutions. However, the use of symmetries is very different in these two methods. The resulting linear equations are therefore also completely different.

For the sake of completeness we note that 'hidden' symmetries, recursion operators and infinite hierarchies of the self-dual-gravity equations were also studied in [22, 23].

11. Conclusion

We have shown that a class of non-invariant solutions of CMA_2 can be obtained by solving a set of linear partial differential equations with constant coefficients for a single real potential. This has enabled us to obtain explicit expressions for the Legendre transform of hyper-Kähler metrics with anti-self-dual Riemann curvature 2-form that admit no continuous symmetries. From the anti-self-duality property it follows that their first Chern class vanishes and the Einstein field equations with Euclidean signature are satisfied.

We started with the extension of Mason–Newman Lax equations for the complex potential of CMA_2 and found that the equation determining the symmetry characteristics of CMA_2 appeared as their integrability condition which led us to the concept of partner symmetries. The use of partner symmetries proved to be the basic tool for finding non-invariant solutions of CMA_2 by solving linear equations.

We established the relation between our concept of partner symmetries and invariance of solutions of CMA_2 with respect to such nonlocal symmetries that CMA_2 itself becomes a consequence of this invariance. This seems to be the spirit of the idea of using 'hidden' symmetries for finding non-invariant solutions of the second heavenly equation suggested in the recent work of Dunajski and Mason. However, our use of symmetries and linear equations determining non-invariant solutions of CMA_2 is completely different.

In this paper, we used the translational and dilatational symmetries of CMA_2 as partner symmetries. We plan to use other pairs of partner symmetries in a future publication in order to obtain further classes of non-invariant solutions of CMA_2 and the corresponding metrics without continuous symmetries.

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